EFFICIENT INVERSE KINEMATICS FOR SERIAL CONNECTIONS OF SERIAL AND PARALLEL MANIPULATORS

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Abstract: This paper presents a unified and efficient method of solving the inverse kinematics for 6 d.o.f. serial connections of two 3 d.o.f. serial and parallel manipulators. Based on the kinematic analysis of 3 d.o. f. serial and parallel manipulators, the forward kinematics are formulated for 6 d.o. f. serial connections, including parallel-parallel, serial-parallel, parallel-serial, and serial-serial types. A unified method of solving the inverse kinematics for the serial connections is established: For a given task velocity, the inverse kinematics of two arms are obtained separately, thereby significantly reducing a computational cost. This is made possible by projecting a given velocity onto the infeasible velocity space of one arm and solving the inverse kinematics of the other arm independently. Through computational analysis, the proposed method is shown to be effective for the serial connections having at least one parallel manipulator.

Introduction

Most manipulators designed up to date are of the form of either serially or parallelly connected kinematic chain. Serial manipulators, such as PUMA, are robotic mechanisms which consist of a series of active revolute or prismatic joints connecting the base to the end-effector, and they are able to achieve large workspace and high dexterity. Parallel manipulators [1-6], such as the Stewart Platform, are robotic mechanisms which consist of a set of parallel limbs, each having one active joint as well as passive joints required to maintain system mobility and controllability and connecting the base plate to the moving plate, and they are able to achieve high stiffness and high force-to-weight ratio.

Several efforts have been made to incorporate the merits of serial and parallel manipulators by connecting them in serial. The serial connections of serial and parallel manipulators can be categorized into the following four types:

- Parallel-Parallel type in which two parallel arms are connected in serial, as shown in Fig.1a. This type of serial connection was proposed in [7] to implement light weight, stiff, and fast-moving manipulator systems.

- Serial-Parallel type in which a parallel arm is mounted on a serial arm, as shown in Fig.1b. This type of serial connection was proposed in [8] to implement a micro manipulator/wrist system with high bandwidth and high resolution.

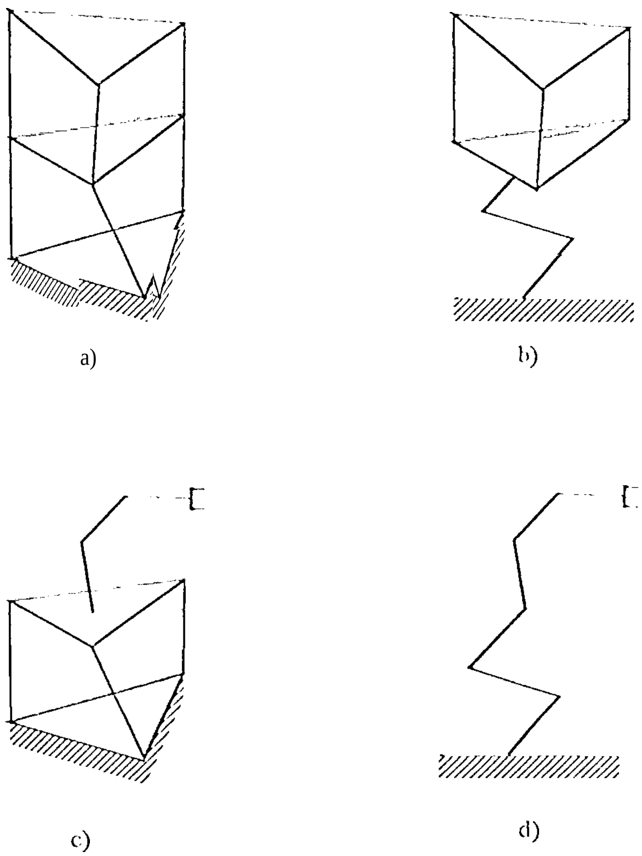


Figure 1: Four types of serial connections: a) parallel-parallel, b) serial-parallel, c) parallel-serial, and d) serial-serial.

- Parallel-Serial type in which a serial arm is mounted on a parallel arm, as shown in Fig.1c. This type of serial connection is reverse to the previous type, and may be useful to enhance the stability and strength of the lower part of the serial connection.

- Serial-Serial type in which two serial arms are connected in serial, as shown in Fig.1d. This type of serial connection can be found in conventional serial arms decomposed at an intermediate point.

To put the serial connections of serial and parallel manipulators into practice, it is important to develop the efficient kinematics of the serial connections. It is well known that for serial manipulators, the forward kinematics is straightforward but the inverse kinematics is complicated, whereas the opposite is true for parallel manipulators. This is due to the fact that there exists the duality between the screw and wrench axes [9]. However, little attention has been made to explore the kinematics of the serial connections of serial and parallel manipulators, characterized by the kinematic features of both serial and parallel manipulators. Waldron et. al. [8] derived both position and velocity kinematics for the serial-parallel type, and Shahinpoor [7] derived position kinematics for the parallel-parallel type.

This paper presents a unified and efficient way of solving the inverse kinematics for 6 d.o.f. serial connections of two 3 d.o.f. serial and parallel manipulators. This paper is organized as follows: In Section 2, we derive the forward and inverse kinematics of a 3 d.o.f. serial and parallel manipulators in 6 dimensional task space. In Section 3, we establish a unified method of solving the inverse kinematics of the serial connections and obtain the efficient inverse kinematics solutions for the serial connections having at least one parallel arm.

For the analysis of computational cost associated with the kinematics solutions, we calculate the number of multiplications, ignoring additions involved. For notational convenience, we define the following:

$C_m(l \times m \times n)$ or $C_m(l \times m) \triangleq$ the cost for multiplying an $1 \times m$ matrix and an $m \times n$ matrix or multiplying an $1 \times m$ matrix and an $m \times 1$ vector.

$C_i(n) \triangleq$ the cost for inverting an $n \times n$ matrix.

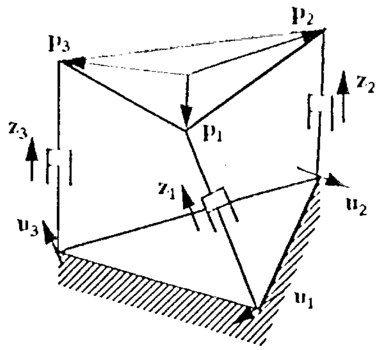
$C_s(m \times n)$ or $C_s(n) \triangleq$ the cost for solving m linear equations with n unknowns or solving n linear equations with n unknowns.

Kinematics of 3 D.O.F. Arm

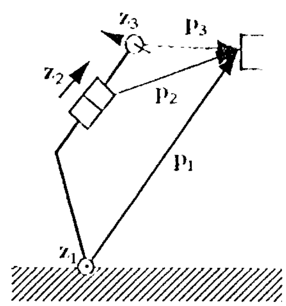
In this section, we derive the forward and inverse kinematics of 3 d.o.f. parallel and serial arms working in 6 dimensional task space. It is assumed for simplicity that a serial arm has revolute joints only and a parallel has prismatic joints only. However, both a serial arm with prismatic and revolute joints and a parallel arm with revolute joints can be treated within the same framework as in this paper [9].

First, consider a 3 d.o.f. parallel arm [2,4,8] consisting of three limbs, each having one active prismatic joint, with its upper end connected to the base plate through a passive pin joint and its lower end connected to the moving plate through a passive ball joint, as shown in Fig.2a. In Fig.2a, for limb k, $k = 1,2,3$, d_k is the position of the prismatic joint, $z_k \in R^{3\times1}$ is a unit vector along the positional axis of the prismatic joint, $u_k \in R^{3\times1}$ is a unit vector along the rotational axis of the pin joint, which is normal to z_k, and $p_k \in R^{3\times1}$ is the position vector from the end-effector on the moving plate to the ball joint. In what follows, all position and velocity vectors of an arm are assumed to be expressed with respect to the base frame fixed to the base plate.

With $v \in R^{3\times1}$ and $w \in R^{3\times1}$ being the linear and angular velocities at the end-effector, the linear velocity at the



a)



b)

Figure 2: a) 3 d.o.f. parallel arm and b) 3 d.o.f. serial arm.

upper end of limb k, $v_k \in R^{3\times1}$, $k = 1,2,3$, is obtained by

$$v_k = v + w \times p_k. \quad (1)$$

Since a parallel arm under consideration is constructed in such a way that v_k, $k = 1,2,3$, is allowed only in the direction of z_k and should lie on the plane perpendicular to u_k, we have [8]

$$z_k \cdot v_k = \dot{d}_k, \quad (2)$$
$$u_k \cdot v_k = 0. \quad (3)$$

Plugging (1) into (2) and (3), and with $\dot{d} \triangleq [\,\dot{d}_1 \;\; \dot{d}_2 \;\; \dot{d}_3\,]^t \in R^{3\times1}$ and $\dot{x} \triangleq [\,v^t \;\; w^t\,]^t \in R^{6\times1}$,

$$B_z \dot{x} = \dot{d}, \quad (4)$$
$$B_u \dot{x} = 0, \quad (5)$$

where

$$B_z = \begin{bmatrix} z_1^t & (p_1 \times z_1)^t \\ z_2^t & (p_2 \times z_2)^t \\ z_3^t & (p_3 \times z_3)^t \end{bmatrix} \in R^{3\times6}, \quad B_u = \begin{bmatrix} u_1^t & (p_1 \times u_1)^t \\ u_2^t & (p_2 \times u_2)^t \\ u_3^t & (p_3 \times u_3)^t \end{bmatrix} \in R^{3\times6}. \quad (6)$$

(4) represents the inverse kinematics of a parallel arm, while (5) represents the velocity constraint due to its construction. It should be noted that (4) and (5) together describe the motion of a parallel arm.

From (5), the feasible velocity space of a parallel arm is given by the nullspace of $\mathbf{B}_u$, $\mathcal{N}(\mathbf{B}_u)$:

$$\dot{\mathbf{x}} \in \mathcal{N}(\mathbf{B}_u),$$

which is 3-dimensional. Combining (4) and (5), we have

$$\begin{bmatrix} \mathbf{B}_z \\ \mathbf{B}_u \end{bmatrix} \dot{\mathbf{x}} = \begin{bmatrix} \dot{\mathbf{d}} \\ 0 \end{bmatrix}. \tag{7}$$

The forward kinematic solution of a parallel arm can be obtained by solving (7), which requires the computational cost of $C_s(6)$.

From (7), the forward kinematics of a parallel arm can be represented as

$$\dot{\mathbf{x}} = \mathbf{A}\dot{\mathbf{d}}, \tag{8}$$

where $\mathbf{A} \in \mathbb{R}^{6\times3}$ is the submatrix of $\begin{bmatrix} \mathbf{B}_z \\ \mathbf{B}_u \end{bmatrix}^{-1}$ corresponding to $\dot{\mathbf{d}}$. In (8), $\mathbf{A}$ can be considered as the Jacobian matrix of a parallel arm, mapping the joint space velocity, $\dot{\mathbf{d}}$, to the task space velocity, $\dot{\mathbf{x}}$.

Next, consider a 3 d.o.f. serial arm consisting of three revolute joints, as shown in Fig.2b, where for joint k, $k = 1, 2, 3$, θ_k is the joint angle, $\mathbf{z}_k \in \mathbb{R}^{3\times1}$ is a unit vector along the rotational axis of the joint, and $\mathbf{p}_k \in \mathbb{R}^{3\times1}$ is the position vector from the joint to the end-effector.

The linear and angular velocities at the end-effector, $\mathbf{v} \in \mathbb{R}^{3\times1}$ and $\mathbf{w} \in \mathbb{R}^{3\times1}$, generated from the joint velocity, $\dot{\theta} \triangleq [\dot{\theta}_1 \;\; \dot{\theta}_2 \; \dot{\theta}_3]^t \in \mathbb{R}^{3\times1}$, are given by [9]

$$\mathbf{v} = \mathbf{J}_v\dot{\theta}, \tag{9}$$
$$\mathbf{w} = \mathbf{J}_w\dot{\theta}, \tag{10}$$

where

$$\begin{aligned} \mathbf{J}_v &= \begin{bmatrix} \mathbf{z}_1 \times \mathbf{p}_1 & \mathbf{z}_2 \times \mathbf{p}_2 & \mathbf{z}_3 \times \mathbf{p}_3 \end{bmatrix} \in \mathbb{R}^{3\times3}, \\ \mathbf{J}_w &= \begin{bmatrix} \mathbf{z}_1 & \mathbf{z}_2 & \mathbf{z}_3 \end{bmatrix} \in \mathbb{R}^{3\times3}. \end{aligned} \tag{11}$$

With $\dot{\mathbf{x}} \triangleq [\mathbf{v}^t \;\; \mathbf{w}^t]^t \in \mathbb{R}^{6\times1}$ and $\mathbf{J} \triangleq [\mathbf{J}_v^t \; \mathbf{J}_w^t]^t \in \mathbb{R}^{6\times3}$, the forward kinematics of a serial arm is obtained by

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\theta}. \tag{12}$$

The feasible velocity space of a serial arm is given by the columnspace of the Jacobian matrix $\mathbf{J}$, $\mathcal{C}(\mathbf{J})$:

$$\dot{\mathbf{x}} \in \mathcal{C}(\mathbf{J}). \tag{13}$$

The inverse kinematic solution of a serial arm can be obtained by solving (12), which requires the computational cost of $C_s(6\times3)$. For a given $\dot{\mathbf{x}} \notin \mathcal{C}(\mathbf{J})$, the inverse kinematic solution represents the least squares solution of (12), which is given by $\dot{\theta} = (\mathbf{J}^t\mathbf{J})^{-1}\mathbf{J}^t\dot{\mathbf{x}}$.

Inverse Kinematics of Serial Connections

This section develops an efficient method of obtaining the inverse kinematics for the serial connections of two 3 d.o.f. arms, either serial or parallel arm. The efficiency is achieved by solving the inverse kinematics of two arms separately, instead of solving the inverse kinematics of the serial connection at a time.

Methodology

The velocity at the end-effector of the serial connection, $\dot{\mathbf{x}}_o \in \mathbb{R}^{6\times1}$, is given by

$$\dot{\mathbf{x}}_o = \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2, \tag{14}$$

where $\dot{\mathbf{x}}_i = [\mathbf{v}_i^t \;\; \mathbf{w}_i^t]^t \in \mathbb{R}^{6\times1}$, $i = 1, 2$, is the velocity of arm i at the end-effector of the serial connection. (14) represents the forward kinematics of the serial connection, and the feasible velocity space of the serial connection is given by

$$\dot{\mathbf{x}}_o \in (\mathcal{V}_1 \oplus \mathcal{V}_2),$$

where $\mathcal{V}_i$, $i = 1, 2$, is the feasible velocity space of arm i and '$\oplus$' represents the space addition. Assuming that the serial connection of two arms has 6 d.o.f.,

$$\begin{aligned} \mathcal{V}_1 \oplus \mathcal{V}_2 &= \mathbb{R}^6, \\ \mathcal{V}_1 \cap \mathcal{V}_2 &= \emptyset, \end{aligned} \tag{15}$$

where '$\cap$' represents the space intersection

The inverse kinematics of the serial connection of two arms may be obtained by considering the inverse kinematics of both arms simultaneously, referred to here as the *direct method*. However, the computational efficiency may be

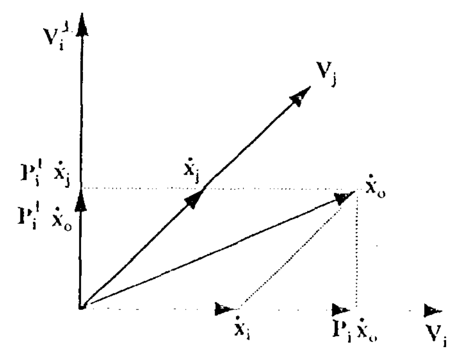


Figure 3: The decomposition of a given task velocity.

achieved by separating the inverse kinematics of the serial connection of two arms into the inverse kinematics of individual arms, as follows:

Let $\mathbf{P}_i$ be the projection operator onto $\mathcal{V}_i$, and $\mathbf{P}_i^\perp$ be the projection operator onto the orthogonal complement of $\mathcal{V}_i$, $\mathcal{V}_i^\perp$. A given task velocity, $\dot{\mathbf{x}}_o$, can be decomposed into two orthogonal components:

$$\dot{\mathbf{x}}_o = \mathbf{P}_i\dot{\mathbf{x}}_o + \mathbf{P}_i^\perp\dot{\mathbf{x}}_o, \tag{16}$$

where $\mathbf{P}_i\dot{\mathbf{x}}_o$ and $\mathbf{P}_i^\perp\dot{\mathbf{x}}_o$ are the components of $\dot{\mathbf{x}}_o$ lying respectively on $\mathcal{V}_i$ and $\mathcal{V}_i^\perp$. This is illustrated in Fig.3. In (16), $\mathbf{P}_i^\perp\dot{\mathbf{x}}_o$ can be generated by arm j only, while $\mathbf{P}_i\dot{\mathbf{x}}_o$ can be generated by arm i or arm j.

Under the assumption of (15), the projection of $\dot{x}_j$ onto $\mathcal{V}_i^\perp$, $P_i^\perp \dot{x}_j$, should be the same as $P_i^\perp \dot{x}_o$:

$$P_i^\perp \dot{x}_o = P_i^\perp \dot{x}_j. \tag{17}$$

Otherwise, $\dot{x}_o \notin (\mathcal{V}_i \oplus \mathcal{V}_2)$. Note that (17) can be also obtained by premultiplying (14) by $P_i^\perp$ and using $P_i^\perp \dot{x}_i = 0$.

Since (17) contains $\dot{x}_j$ only, the inverse kinematics of arm j can be obtained first, independently of arm i. With the inverse kinematics of arm j known, from (14),

$$\dot{x}_i = \dot{x}_o - \dot{x}_j. \tag{18}$$

which can be used to obtain the inverse kinematics of arm i.

The method of solving the inverse kinematics of the serial connection based on (17) and (18), is referred to here as the *projection method.* The success of the projection method depends on

- the availability of the projection operator,
- the availability of the forward kinematics of individual arms.

Note that serial and parallel arms are reciprocal to each other in terms of the availability of the projection operator and the forward kinematics.

Parallel-Parallel Type (P-P)

For the serial connection of two parallel arms shown in Fig.1a, the velocity of arm i, $\dot{x}_i, i=1,2$, is given by

$$\begin{bmatrix} B_{iz} \\ B_{iu} \end{bmatrix} \dot{x}_i = \begin{bmatrix} \dot{d}_i \\ 0 \end{bmatrix}, \tag{19}$$

where $\dot{d}_i \in \mathbb{R}^{3\times 1}$ is the joint velocity; $B_{iz} \in \mathbb{R}^{3\times 6}$ and $B_{iu} \in \mathbb{R}^{3\times 6}$ are given in the form of (6), represented at the end-effector of the serial connection.

The feasible velocity space of the parallel-parallel type is given by $[\mathcal{N}(B_{1u}) \oplus \mathcal{N}(B_{2u})]$, and its forward kinematic solution can be obtained by solving (19) and using (14), which requires the approximated computational cost of $2 \cdot C_s(6)$.

With

$$\dot{x}_i = A_i \dot{d}_i,$$

where $A_i \in \mathbb{R}^{6\times 3}$ is the submatrix of $\begin{bmatrix} B_{iz} \\ B_{iu} \end{bmatrix}^{-1}$ corresponding to $\dot{d}_i$, the forward kinematics of the parallel-parallel type is given by

$$\dot{x}_o = [A_1 \quad A_2] \begin{bmatrix} \dot{d}_1 \\ \dot{d}_2 \end{bmatrix}. \tag{20}$$

Based on (20), the inverse kinematic solution of the parallel-parallel type can be obtained in the following steps:

Step 1: Compute $\begin{bmatrix} B_{iz} \\ B_{iu} \end{bmatrix}^{-1}$ and construct A_i, $i = 1, 2$.

Step 2: Compute $\dot{d}_1$ and $\dot{d}_2$ by solving (20).

The computational cost associated with the above solution can be approximated as the sum of $2 \cdot C_i(6)$ for Step 1 and $C_s(6)$ for Step 2, which amounts to

$$2 \cdot C_i(6) + C_s(6).$$

Now, based on the projection method, let us consider the inverse kinematics of the serial connection of two parallel arms as follows:

Premultipled by B_{1u}, (14) becomes

$$B_{1u}\dot{x}_o = B_{1u}\dot{x}_2, \tag{21}$$

since $B_{1u}\dot{x}_1 = 0$. Using $\dot{x}_2 = A_2\dot{d}_2$, we have

$$B_{1u}\dot{x}_o = (B_{1u}A_2)\dot{d}_2. \tag{22}$$

With $\dot{d}_2$ known, $\dot{d}_1$ is obtained by

$$\dot{d}_1 = B_{1z}(\dot{x}_o - A_2\dot{d}_2). \tag{23}$$

Based on (22) and (23), the inverse kinematic solution of the parallel-parallel type can be obtained in the following Steps:

Step 1: Compute $\begin{bmatrix} B_{2z} \\ B_{2u} \end{bmatrix}^{-1}$ and construct A_2.

Step 2: Compute $\dot{d}_2$ by solving (22).

Step 3: With $\dot{d}_2$ known, compute $\dot{d}_1$ using (23).

The computational cost associated with the above can be approximated as the sum of $C_i(6)$ for Step 1, $[C_m(3\times 6\times 3) + C_m(3\times 6) + C_s(3)]$ for Step 2, and $[C_m(6\times 3) + C_m(3\times 6)]$ for Step 3, which amounts to

$$C_i(6) + C_m(3\times 6\times 3) + 3\cdot C_m(3\times 6) + C_s(3).$$

Serial-Parallel Type (S-P)

For the serial connection of serial and parallel arms shown in Fig.1b, the velocity of arm i, $\dot{x}_i, i=1,2$, is given by

$$\dot{x}_1 = J_1\dot{\theta}_1, \tag{24}$$

$$\begin{bmatrix} B_{2z} \\ B_{2u} \end{bmatrix} \dot{x}_2 = \begin{bmatrix} \dot{d}_2 \\ 0 \end{bmatrix}, \tag{25}$$

where $\dot{\theta}_1 \in \mathbb{R}^{3\times 1}$ is the joint velocity; $J_1 \in \mathbb{R}^{6\times 3}$ is the Jacobian matrix of arm 1, represented at the end-effector of the serial connection.

The feasible velocity spaces of the serial-parallel type is given by $[\mathcal{C}(J_1) \oplus \mathcal{N}(B_{2u})]$, and its forward kinematic solution can be obtained by solving (25) and using (24) and (14), which requires the approximated computational cost of $[C_m(6\times 3) + C_s(6)]$.

With $\dot{\mathbf{x}}_2 = \mathbf{A}_2\dot{\mathbf{d}}_2$, the forward kinematics of the serial-parallel type is written as

$$\dot{\mathbf{x}}_o = \begin{bmatrix} \mathbf{J}_1 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\mathbf{d}}_2 \end{bmatrix} \tag{26}$$

Based on (26), the inverse kinematic solution of the serial-parallel type can be obtain in the following steps:

Step 1: Compute $\mathbf{A}_2$.

Step 2: Compute $\dot{\theta}_1$ and $\dot{\mathbf{d}}_2$ by solving (26).

The computational cost associated with the above solution can be approximated as the sum of $C_i(6)$ for Step 1 and $C_s(6)$ for Step 2:

$$C_i(6) + C_s(6).$$

Now, based on the projection method, let us consider the inverse kinematics of the serial connection of serial and parallel arms as follows:

Premultiplied by $\mathbf{B}_{2u}$, (14) can be written as

$$\mathbf{B}_{2u}\dot{\mathbf{x}}_o = (\mathbf{B}_{2u}\mathbf{J}_1)\dot{\theta}_1, \tag{27}$$

using $\mathbf{B}_{2u}\dot{\mathbf{x}}_2 = \mathbf{0}$ and $\dot{\mathbf{x}}_1 = \mathbf{J}_1\dot{\theta}_1$. Note that an expression similar to (27) was also obtained by Waldron et. al for the ARTISAN manipulator [8], although their derivation is specific to their system.

With $\dot{\theta}_1$ known, $\dot{\mathbf{d}}_2$ is obtained by

$$\dot{\mathbf{d}}_2 = \mathbf{B}_{2z}(\dot{\mathbf{x}}_o - \mathbf{J}_1\dot{\theta}_1). \tag{28}$$

Based on (27) and (28), the inverse kinematic solution of the serial-parallel type can be obtained in the following steps:

Step 1: Compute $\dot{\theta}_1$ solving (27)

Step 2: With $\dot{\theta}_1$ known, compute $\dot{\mathbf{d}}_2$ using (28).

The computational cost associated with the above solution can be approximated as the sum of $[C_m(3\times 6 \times 3) + C_m(3 \times 6) + C_s(3)]$ for Step 1 and $[C_m(6 \times 3) + C_m(3 \times 6)]$ for Step 2:

$$C_m(3 \times 6 \times 3) + 3 \cdot C_m(3 \times 6) + C_s(3).$$

Parallel-Serial Type (P-S)

For the serial connection of parallel and serial arms shown in Fig.1c, the velocity of arm i, $\dot{\mathbf{x}}_i$, $i = 1, 2$, is given by

$$\begin{bmatrix} \mathbf{B}_{1z} \\ \mathbf{B}_{1u} \end{bmatrix} \dot{\mathbf{x}}_1 = \begin{bmatrix} \dot{\mathbf{d}}_1 \\ \mathbf{0} \end{bmatrix}, \tag{29}$$

$$\dot{\mathbf{x}}_2 = \mathbf{J}_2\dot{\theta}_2. \tag{30}$$

Note that the parallel-serial type is dual to the serial-parallel type in terms of kinematics.

The feasible velocity spaces of the parallel-serial type is given by $[\mathcal{N}(\mathbf{B}_{1u}) \oplus \mathcal{C}(\mathbf{J}_2)]$, and its forward kinematic solution can be obtained by solving (29) and using (30) and (14), which requires the approximated computational cost of $[C_m(6\times 3) + C_s(6)]$.

The inverse kinematic solution of the parallel-serial type can be obtained based on

$$\mathbf{B}_{1u}\dot{\mathbf{x}}_o = (\mathbf{B}_{1u}\mathbf{J}_2)\dot{\theta}_2,$$
$$\dot{\mathbf{d}}_1 = \mathbf{B}_{1z}(\dot{\mathbf{x}}_o - \mathbf{J}_2\dot{\theta}_2),$$

and its associated computational cost can be approximated as $[C_m(3\times 6 \times 3) + 3 \cdot C_m(3 \times 6) + C_s(3)]$.

Serial-Serial Type (S-S)

For the serial connection of two serial arms shown in Fig.1d, the velocity of arm i, $\dot{\mathbf{x}}_i$, $i = 1, 2$, is given by

$$\dot{\mathbf{x}}_i = \mathbf{J}_i\dot{\theta}_i. \tag{31}$$

The feasible velocity space of the serial-serial type is given by $[\mathcal{C}(\mathbf{J}_1) \oplus \mathcal{C}(\mathbf{J}_2)]$, and its forward kinematic solution can be obtained by using (31) and (14), which requires the approximated computation cost of $2 \cdot C_m(6 \times 3)$.

Plugging (31) into (14), the forward kinematics of the serial-serial type is written as

$$\dot{\mathbf{x}}_o = \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \tag{32}$$

Note that (32) can be considered as the forward kinematics of a single 6 d.o.f. serial arm with arm 1 and arm 2 as its lower and upper parts. The inverse kinematic solution of the Serial-serial type is obtained by solving (32), which requires the computational cost of $C_s(6)$.

Now, let us consider the inverse kinematics of the serial connection of two serial arms based on the projection method as follows:

Plugging (31) into (17) and (18), the inverse kinematic solution of the serial-serial type can be obtained by solving

$$\mathbf{P}_1^{\perp}\dot{\mathbf{x}}_o = (\mathbf{P}_1^{\perp}\mathbf{J}_2)\dot{\theta}_2, \tag{33}$$

$$\mathbf{J}_1\dot{\theta}_1 = \dot{\mathbf{x}}_o - \mathbf{J}_2\dot{\theta}_2. \tag{34}$$

where $\mathbf{P}_1^{\perp} \in \mathbb{R}^{3\times 6}$ is the projection operator onto $\mathcal{N}(\mathbf{J}_1^{t})$. Note that for the serial-serial type, the projection operator is not readily available but needs to be computed, while the forward kinematics is readily available. The computational cost associated with the above can be approximated by the sum of $[C_m(3\times 6\times 3) + C_m(3\times 6) + C_s(3)]$ for solving (33) and $[C_m(6 \times 3) + C_s(6 \times 3)]$ for solving (34), as well as $C_s(6 \times 3)$ for computing $\mathbf{P}_1^{\perp}$, which amounts to

$$C_m(3\times 6 \times 3) + 2 \cdot C_m(3 \times 6) + C_s(3) + 2 \cdot C_s(6 \times 3).$$

Using the Gaussian elimination method, the computational cost of obtaining $\mathbf{P}_1^{\perp}$ may be roughly approximated as $C_s(6 \times 3)$.

Thus far, the inverse kinematics has been discussed for a general form of serial-serial type. It is interesting to consider the inverse kinematics of the serial-serial type with special geometry, in which certain three components of a given task velocity can be generated or affected by certain three joints. For instance, with any three co-intersecting revolute joints, the linear velocity at the co intersecting point is independent of the three co intersecting revolute joints [10]. Other 6 d.o.f. serial arms with special geometry include an arm with three prismatic joints, an arm with two prismatic joints normal to a revolute joint, etc.

For the serial-serial type with special geometry, the forward kinematics can be obtained, through appropriate rotation and shuffling operations, in the form of

$$\begin{bmatrix} \hat{\dot{\mathbf{x}}}_{o1} \\ \hat{\dot{\mathbf{x}}}_{o2} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{J}}_{11} & 0_3 \\ \hat{\mathbf{J}}_{21} & \hat{\mathbf{J}}_{22} \end{bmatrix} \begin{bmatrix} \hat{\dot{\theta}}_1 \\ \hat{\dot{\theta}}_2 \end{bmatrix}, \tag{35}$$

where $\left[\hat{\dot{\mathbf{x}}}_{o1}^t \ \ \hat{\dot{\mathbf{x}}}_{o2}^t\right]^t$ is the modified task velocity and $\left[\hat{\dot{\theta}}_1^t \ \ \hat{\dot{\theta}}_2^t\right]^t$ is the shuffled joint velocity.

Based on (35), the inverse kinematic solution can be obtained by solving

$$\hat{\dot{\mathbf{x}}}_{o1} = \hat{\mathbf{J}}_{11}\hat{\dot{\theta}}_1, \tag{36}$$

$$\hat{\dot{\mathbf{x}}}_{o2} - \mathbf{J}_{21}\,\theta_1 = \hat{\mathbf{J}}_{22}\hat{\dot{\theta}}_2, \tag{37}$$

which is referred here as the *decomposition method*. The computational cost can be approximated as the sum of $C_m(6 \times 6)$ for computing $\hat{\dot{\mathbf{x}}}_{o1}$ and $\hat{\dot{\mathbf{x}}}_{o2}$, $C_s(3)$ for solving (36), and $[C_m(3\times 3) + C_s(3)]$ for solving (37), which amounts to

$$C_m(6 \times 6) + C_m(3 \times 3) + 2 \cdot C_s(3).$$

It is worth observing that there exists a certain relationship between the decomposition method and the projection method. The decomposition method is based on the fact that there exists a subset of a given task velocity, which can be generated by a subset of joints. Whereas, the projection method is based on the fact that that there exists a subset of a given task velocity, which cannot be generated by a subset of joints. Note that the decomposition method is effective for a serial arm with special geometry, while the projection method is effective for the serial connections having at least one parallel arm.

Comparison

Let us summarize the computational cost associated with the forward and inverse kinematic solutions of four types of serial connections. To provide numerical comparison among solutions, we count the number of multiplications, using the following approximation [11]:

$$C_i(n) \simeq n^3,$$

$$C_s(m \times n) \simeq \frac{1}{2}mn^2 - \frac{1}{6}n^3 + \frac{1}{2}n^2,$$

where Gaussian elimination and backsubstitution is assumed to be used.

First, the computational cost associated with the forward kinematic solutions of the serial connections is summarized as follows:

type	the number of multiplications
P-P	$2 \cdot C_s(6) \simeq 180$
S-P or P-S	$C_m(6 \times 3) + C_s(6) \simeq 108$
S-S	$2 \cdot C_m(6 \times 3) \simeq 36$

From the table shown above, the computational cost of the forward kinematics increases in the order of serial-serial, serial-parallel (or parallel-serial), and parallel-parallel types. This reflects the fact that the forward kinematic solution of a serial arm can be straightforwardly obtained at a lower computational cost of $C_m(6 \times 3)$, while the forward kinematic solution of a parallel arm can be obtained at a far higher computational cost of $C_s(6)$.

Second, the computational cost associated with the inverse kinematic solutions of the serial connections based on the direct method is summarized as follows:

type	the number of multiplications
P-P	$2 \cdot C_i(6) + C_s(6) \simeq 522$
S-P or P-S	$C_i(6) + C_s(6) \simeq 306$
S-S	$C_s(6) \simeq 90$

From the table shown above, the computational cost of the inverse kinematics based on the direct method increases in the order of serial-serial, serial-parallel (or parallel-serial), and parallel-parallel types. This is because the Jacobian matrix of a serial arm can be readily available, but the Jacobian matrix of a parallel arm needs to be computed at the computational cost of $C_i(6)$.

Third, the computational cost associated with the inverse kinematic solutions of the serial connections based on the projection method is summarized as follows:

type	the number of multiplications
P-P	$C_i(6) + C_m(3\times 6 \times 3) + 3 \cdot C_m(3 \times 6) + C_s(3)$ $\simeq 338$
S-P or P-S	$C_m(3 \times 6 \times 3) + 3 \cdot C_m(3 \times 6) + C_s(3)$ $\simeq 122$
S-S	$C_m(3 \times 6 \times 3) + 2 \cdot C_m(3 \times 6) + C_s(3) + 2 \cdot C_s(6 \times 3)$ $\simeq 158$

The above table shows that the projection method is more efficient than the direct method, for solving the inverse kinematics of the serial connections having at least one parallel arm. This is due to the fact that the projection operator is readily provided by a parallel arm and the Jacobian matrix is made readily available by a serial arm. Note that the projection method does not provide any computational gain for the serial-serial type, and the decomposition method may be considered to be effective for the serial-serial type with special geometry, requiring the computational cost of $[C_m(6 \times 6) + C_m(3 \times 3) + 2 \cdot C_s(3) \simeq 73]$.

Conclusion

A unified and efficient method, referred to as the projection method, was proposed for solving the inverse kinematics for 6 d.o.f. serial connections of two 3 d.o.f. serial and parallel manipulators. The proposed method projects a given task velocity onto the infeasible motion space of one arm, and thus separates the inverse kinematics of the other arm from the inverse kinematics of the serial connection. The success of the proposed method depends on the availability of the projection operator and the forward kinematics of two arms. The computational efficiency of the projection method is shown to be significant for the serial connections having at least one parallel manipulator.

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